

Exponential decay of correlations for a real valued dynamical system embedded in \mathbb{R}^2

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Abstract

We study the real valued process $\{X_t, t \in \mathbb{N}\}$ defined by $X_{t+2} = \varphi(X_t, X_{t+1})$, where the X_t are bounded. We aim at proving the decay of correlations for this model, under regularity assumptions on the transformation φ .

1 Introduction

Since the eighties, the study by statisticians of nonlinear time series has allowed to model a great number of phenomena in Physics, Economics and Finance. But in the nineties the theory of Chaos became an essential axis of research for the study of these processes. For an exhaustive review on this subject, one can consult Collet-Eckmann [CE] about chaos theory and Chan-Tong [TON1, TON2] about nonlinear time series. Within this framework, a general model could be written as

$$X_{t+1} = \varphi(X_t, \dots, X_{t-d+1}) + \varepsilon_t,$$

where φ is nonlinear and ε_t is a noise. We propose a first study of the “skeleton” of this model, as Tong calls it, beginning with $d = 2$ and, more precisely, of the dynamical system induced by this model. Indeed, we consider the model with bounded variables, $X_{t+2} = \varphi(X_t, X_{t+1})$, with $\varphi : \mathcal{U}^2 \rightarrow \mathcal{U}$ for $\mathcal{U} = [-L, L]$ and $L \in \mathbb{R}_+^*$, φ being defined piecewise on \mathcal{U}^2 . This model gives rise to a dynamical system (Ω, τ, μ, T) where μ is a measure, invariant under the transformation $T : \Omega \rightarrow \Omega$ and Ω is a compact subset of \mathbb{R}^2 . Under hypotheses on φ , which imply that T satisfies the hypotheses of Saussol [SAU], and if we suppose that T is mixing, we obtain the exponential decay of correlations. More precisely, for well-chosen applications f and h , there exist constants $C = C(f, h) > 0$, $0 < \rho < 1$ such that:

$$\left| \int_{\Omega} f \circ T^k h \, d\mu - \int_{\Omega} f \, d\mu \int_{\Omega} h \, d\mu \right| \leq C \rho^k.$$

This result yields a covariance inequality of the following kind:

$$|\text{Cov}(f(X_k), h(X_0))| \leq C \rho^k.$$

Other ways could certainly be used to get the same result, under different hypotheses on the induced system, for example the method of Young towers [YOU]. To have a general

view on these different technics, one can read the article of Alves-Freitas-Luzzato-Vaianti [AFLV].

We finish by giving two examples illustrating our results, a piecewise linear one and a nonlinear one.

2 Hypotheses and results

Let $L \in \mathbb{R}_+^*$. Let $\varphi : [-L, L]^2 \rightarrow [-L, L]^2$ be piecewise defined on $[-L, L]^2$. To study the process $\{X_t, t \in \mathbb{N}\}$ defined by $X_{t+2} = \varphi(X_t, X_{t+1})$, there exist different ways of choosing the induced dynamical system $Z_{t+1} = T(Z_t)$ with $Z_t \in \mathbb{R}^2$. We tried two different approaches, on the one hand the canonical method, setting $T(x, y) = (y, \varphi(x, y))$ and on the other hand a double iteration, which comes down to setting $T(x, y) = (\varphi(x, y), \varphi(y, \varphi(x, y)))$. The first approach, up to a conjugation, is the most fruitful, the second one requiring stronger hypotheses and yielding weaker results. We therefore set $T(x, y) = (\frac{y}{\gamma}, \gamma\varphi(x, \frac{y}{\gamma}))$ with $Z_t = (X_t, \gamma X_{t+1})$, for a suitable positive γ . It then became possible to work in spaces similar to Saussol's V_α and to use his results.

More precisely, we suppose that the following hypotheses are fulfilled:

(H1) there exists $d \in \mathbb{N}^*$ such that

$$[-L, L]^2 = \bigcup_{k=1}^d O_k \cup \mathcal{N},$$

where the O_k are nonempty open sets, \mathcal{N} is negligible for the Lebesgue measure and the union is disjoint. The edges of the O_k can be split into a finite number of smooth components, each one included in a C^1 , compact and one dimensional submanifold of \mathbb{R}^2 .

(H2) There exists $\varepsilon_1 > 0$ such that, for all $k \in \{1, \dots, d\}$, there exists an application φ_k defined on $B_{\varepsilon_1}(\overline{O_k}) = \{(x, y) \in \mathbb{R}^2, d((x, y), \overline{O_k}) \leq \varepsilon_1\}$, with values in \mathbb{R} , such that $\varphi_k|_{O_k} = \varphi|_{O_k}$.

(H3) The application φ_k is bounded, $C^{1,\alpha}$ on $B_{\varepsilon_1}(\overline{O_k})$ for a real $\alpha \in]0, 1]^2$, which means that φ_k is C^1 and that there exists $C_k > 0$ such that, for all $(u, v), (u', v')$ in $B_{\varepsilon_1}(\overline{O_k})$,

$$\left| \frac{\partial \varphi_k}{\partial u}(u, v) - \frac{\partial \varphi_k}{\partial u}(u', v') \right| \leq C_k \|(u, v) - (u', v')\|^\alpha$$

$$\left| \frac{\partial \varphi_k}{\partial v}(u, v) - \frac{\partial \varphi_k}{\partial v}(u', v') \right| \leq C_k \|(u, v) - (u', v')\|^\alpha.$$

We moreover suppose that there exist $A > 1$ and $M \in]0, A - 1[$ such that :

$$\forall (u, v) \in B_{\varepsilon_1}(\overline{O_k}), \quad \left| \frac{\partial \varphi_k}{\partial u}(u, v) \right| \geq A, \quad \left| \frac{\partial \varphi_k}{\partial v}(u, v) \right| \leq M,$$

to ensure the expansion.

¹ To get similar results on $[a, b]$ instead of $[-L, L]$, it suffices to conjugate by an affine application

² If φ_k is C^2 on $B_{\varepsilon_1}(\overline{O_k})$, it is $C^{1,\alpha}$ on $B_{\varepsilon_1}(\overline{O_k})$ with $\alpha = 1$

(H4) The open sets O_k satisfy the following geometrical condition: ³ For all (u, v) and (u', v) in $B_{\varepsilon_1}(\overline{O_k})$, there exists a C^1 path $\Gamma = (\Gamma_1, \Gamma_2) : [0, 1] \rightarrow B_{\varepsilon_1}(\overline{O_k})$ C^1 joining (u, v) and (u', v) , whose gradient does not vanish and which satisfies

$$\forall t \in]0, 1[, |\Gamma'_1(t)| > \frac{M}{A} |\Gamma'_2(t)|.$$

(H5) Let $Y \in \mathbb{N}^*$ be the maximal number of C^1 components of \mathcal{N} meeting at one point and set

$$s = \left(\frac{2A + M^2 - M\sqrt{M^2 + 4A}}{2} \right)^{-1/2} < 1.$$

One supposes that

$$\eta := s^\alpha + \frac{8s}{\pi(1-s)}Y < 1.$$

We set $\gamma = \frac{1}{\sqrt{A}} < 1$ and, for all $k \in \{1, \dots, d\}$, we denote by U_k (resp. W_k, \mathcal{N}') the image of O_k (resp. $B_{\varepsilon_1}(\overline{O_k}), \mathcal{N}$) under the compression which associates $(u, \gamma v)$ with each $(u, v) \in \mathbb{R}^2$.

The set $\Omega = [-L, L] \times [-\gamma L, \gamma L]$, on which we shall be working, is the image of $[-L, L]^2$ under the same compression.

For every non negligible Borel set S of \mathbb{R}^2 , for every $f \in L_m^1(\mathbb{R}^2, \mathbb{R})$, set

$$Osc(f, S) = E_{sup}^S f - E_{inf}^S f,$$

where E_{sup}^S and E_{inf}^S are the essential supremum and infimum with respect to the Lebesgue measure m . One then defines:

$$|f|_\alpha = \sup_{0 < \varepsilon < \varepsilon_1} \varepsilon^{-\alpha} \int_{\mathbb{R}^2} \text{Osc}(f, B_\varepsilon(x, y)) \, dx dy \quad , \quad \|f\|_\alpha = \|f\|_{L_m^1} + |f|_\alpha$$

and the set $V_\alpha = \{f \in L_m^1(\mathbb{R}^2, \mathbb{R}), \|f\|_\alpha < +\infty\}$.

Let us introduce similar notions on Ω : for every $0 < \varepsilon_0 < \gamma\varepsilon_1$, for every $g \in L_m^\infty(\Omega, \mathbb{R})$, one defines

$$N(g, \alpha, L) = \sup_{0 < \varepsilon < \varepsilon_0} \varepsilon^{-\alpha} \int_{\Omega} \text{Osc}(g, B_\varepsilon(x, y) \cap \Omega) \, dx dy.$$

One then sets:

$$\|g\|_{\alpha, L} = N(g, \alpha, L) + 16(1 + \gamma)\varepsilon_0^{1-\alpha}L\|g\|_\infty + \|g\|_{L_m^1}.$$

The function g is said to belong to $V_\alpha(\Omega)$ if the above expression is finite. The set $V_\alpha(\Omega)$ does not depend on the choice of ε_0 , whereas N and $\|\cdot\|_{\alpha, L}$ do.

There exist relationships between these two sets. Indeed, thanks to Proposition 3.4 of [SAU], one can prove the following result:

³In suitable cases, this hypothese can be replaced by a weaker but simpler one : for all points (u, v) and (u', v) in $B_{\varepsilon_1}(\overline{O_k})$, the segment $[(u, v), (u', v)]$ is included in $B_{\varepsilon_1}(\overline{O_k})$

Proposition 1 1. If $g \in V_\alpha(\Omega)$ and if one extends g as a function denoted by f , setting $f(x, y) = 0$ if $(x, y) \notin \Omega$, then $f \in V_\alpha$ and

$$\|f\|_\alpha \leq \|g\|_{\alpha, L}.$$

2. Let f be in V_α . Set $g = f\mathbf{1}_\Omega$. Then $g \in V_\alpha(\Omega)$ and one has

$$\|g\|_{\alpha, L} \leq \left(1 + 16(1 + \gamma)L \frac{\max(1, \varepsilon_0^\alpha)}{\pi \varepsilon_0^{1+\alpha}}\right) \|f\|_\alpha.$$

Under the above hypotheses (H1) to (H5), one obtains a first result:

Theorem 2 Let T be the transformation defined on Ω by : $\forall (x, y) \in U_k$:

$$T(x, y) = T_k(x, y) = \left(\frac{y}{\gamma}, \gamma \varphi_k(x, \frac{y}{\gamma})\right).$$

Keeping the same formula, one extends the definition of T_k to W_k . Then

1. The Frobenius-Perron operator $P : L_m^1(\Omega) \rightarrow L_m^1(\Omega)$ associated with T has a finite number of eigenvalues $\lambda_1, \dots, \lambda_r$ of modulus one.
2. For each $i \in \{1, \dots, r\}$, the eigenspace $E_i = \{f \in L_m^1(\Omega) : Pf = \lambda_i f\}$ associated with the eigenvalue λ_i is finite dimensional and included in $V_\alpha(\Omega)$.
3. The operator P decomposes as

$$P = \sum_{i=1}^r \lambda_i P_i + Q,$$

where the P_i are projections on the spaces E_i , $\|P_i\|_1 \leq 1$ and Q is a linear operator defined on $L_m^1(\Omega)$, satisfying $Q(V_\alpha(\Omega)) \subset V_\alpha(\Omega)$, $\sup_{n \in \mathbb{N}^*} \|Q^n\|_1 < \infty$ and $\|Q^n\|_{\alpha, L} = O(q^n)$ when $n \rightarrow +\infty$ for an exponent $q \in]0, 1[$. Moreover, $P_i P_j = 0$ if $i \neq j$, $P_i Q = Q P_i = 0$ for all i .

4. The number 1 is an eigenvalue of P . Set $\lambda_1 = 1$, let $h_* = P_1 \mathbf{1}_\Omega$ and let $d\mu = h_* dm$. Then μ is the greatest absolutely continuous invariant measure (ACIM) of T , that is to say: if $\nu \ll m$ and if ν is T -invariant, then $\nu \ll \mu$.
5. The support of μ can be decomposed into a finite number of disjoint measurable sets, on which a power of T is mixing. More precisely for all $j \in \{1, 2, \dots, \dim(E_1)\}$, there exist an integer $L_j \in \mathbb{N}^*$ and L_j disjoint sets $W_{j,l}$ ($0 \leq l \leq L_j - 1$) satisfying $T(W_{j,l}) = W_{j, l+1 \bmod L_j}$ and T^{L_j} is mixing on every $W_{j,l}$. We denote by $\mu_{j,l}$ the normalized restriction of μ to $W_{j,l}$, defined by

$$\mu_{j,l}(B) = \frac{\mu(B \cap W_{j,l})}{\mu(W_{j,l})}, \quad d\mu_{j,l} = \frac{h^* \mathbf{1}_{W_{j,l}}}{\mu(W_{j,l})} dm.$$

The fact that T^{L_j} is mixing on every $W_{j,l}$ means that, for all $f \in L_{\mu_{j,l}}^1(W_{j,l})$ and all $h \in L_{\mu_{j,l}}^\infty(W_{j,l})$,

$$\lim_{t \rightarrow +\infty} \langle T^{tL_j} f, h \rangle_{\mu_{j,l}} = \langle f, 1 \rangle_{\mu_{j,l}} \langle 1, h \rangle_{\mu_{j,l}}$$

with the notations (indifferently employed) $\langle f, g \rangle_{\mu'} = \mu'(fg) = \int fg \, d\mu'$.

6. Moreover, there exist $C > 0$ and $0 < \rho < 1$ such that, for all h in $V_\alpha(\Omega)$ and $f \in L_\mu^1(\Omega)$, one has

$$\left| \int_\Omega f \circ T^{k \times \text{ppcm}(L_i)} h \, d\mu - \sum_{j=1}^{\dim(E_1)} \sum_{l=0}^{L_j-1} \mu(W_{j,l}) \langle f, 1_{>\mu_{j,l}} < 1, h \rangle_{\mu_{j,l}} \right| \leq C \|h\|_{\alpha, \Omega} \|f\|_{L_\mu^1(\Omega)} \rho^k.$$

7. If, moreover, T is mixing⁴, then the preceding result can be written as follows: there exist $C > 0$ and $0 < \rho < 1$ such that, for all h in $V_\alpha(\Omega)$ and $f \in L_\mu^1(\Omega)$, one has:

$$\left| \int_\Omega f \circ T^k h \, d\mu - \int_\Omega f \, d\mu \int_\Omega h \, d\mu \right| \leq C \|h\|_{\alpha, \Omega} \|f\|_{L_\mu^1(\Omega)} \rho^k.$$

Now let us come back to the initial problem and try to deduce from this result the invariant law associated with X_t . If $(X_t)_t$ is defined by X_0, X_1 (valued in $[-L, L]$) and the recurrence relation $X_{t+2} = \varphi(X_t, X_{t+1})$, one sets $Z_t = (X_t, \gamma X_{t+1})$. Then $(Z_t)_t$ satisfies the recurrence relation $Z_{t+1} = T(Z_t)$, which implies the following result:

Theorem 3 Suppose that the random variable $Z_0 = (X_0, \gamma X_1)$ has the density h_* . Then Z_t has the density h_* and for all $t \in \mathbb{N}$, X_t has the density

$$f : x \mapsto \int_{[-\gamma L, \gamma L]} h_*(x, v) \, dv = \gamma \int_{[-L, L]} h_*(u, \gamma x) \, du. \quad (1)$$

Indeed, since $Z_t = (X_t, \gamma X_{t+1})$ has the density h_* , one proves that X_t has the density f by computing the first marginal distribution. Computing the second one yields that γX_{t+1} has the density $g = g(y)$ defined by

$$g(y) = \int_{[-L, L]} h_*(u, y) \, du.$$

This implies that X_{t+1} has the density $y \mapsto \gamma g(\gamma y)$. But Z_{t+1} has the density h_* as well. Therefore X_{t+1} has the density given by the first marginal distribution, which proves the equality (1).

If F is defined on $[-L, L]$, we denote by $Tr F$ the function defined, on Ω , by $Tr F(x, y) = F(x)$.

One then obtains the following result, which is a direct consequence of the sixth point of Theorem 2, applied to $Tr F$ and $Tr H$:

Theorem 4 For every Borel set B and every interval I , if (X_0, X_1) has the invariant distribution, then

$$\left| P(X_{k \times \text{ppcm}(L_i)} \in B, X_0 \in I) - \sum_{j=1}^{\dim(E_1)} \sum_{l=0}^{L_j-1} \mu(W_{j,l}) \langle Tr \mathbf{1}_B, 1_{>\mu_{j,l}} < 1, Tr \mathbf{1}_I \rangle_{\mu_{j,l}} \right| \leq 16(1 + \gamma) C L^3 (10\varepsilon_0^{1-\alpha} + L) \rho^k.$$

⁴ which is equivalent to: if 1 is the only eigenvalue of P with modulus one and if it is simple

More generally, let F , defined and measurable on $[-L, L]$, be such that $\text{Tr } F$ belongs to $L^1_\mu(\Omega)$. Let $H \in L^\infty_m([-L, L])$ be such that $\sup_{0 < \varepsilon < \varepsilon_0} \varepsilon^{-\alpha} \int_{[-L, L]} \text{Osc}(H,]x - \varepsilon, x + \varepsilon[\cap [-L, L]) \, dx < +\infty$.

Then $\text{Tr } H \in V_\alpha(\Omega)$ and

$$\left| E(F(X_{k \times \text{ppcm}(L_i)})H(X_0)) - \sum_{j=1}^{\dim(E_1)} \sum_{l=0}^{L_j-1} \mu(W_{j,l}) \mu_{j,l}(\text{Tr } F) \mu_{j,l}(\text{Tr } H) \right| \leq C(F, H) \, \rho^k$$

with

$$C(F, H) = C \, L \, \|\text{Tr } F\|_{L^1_\mu} \left(2\gamma \sup_{0 < \varepsilon < \varepsilon_0} \varepsilon^{-\alpha} \int_{[-L, L]} \text{Osc}(H,]x - \varepsilon, x + \varepsilon[\cap [-L, L]) \, dx + 16(1 + \gamma) \, \varepsilon_0^{1-\alpha} \|H\|_{L^\infty_m([-L, L])} + 2\gamma \|H\|_{L^1_m([-L, L])} \right).$$

If, moreover, T is mixing, then:

$$|\text{Cov}(F(X_k), H(X_0))| \leq C(F, H) \, \rho^k.$$

3 Proofs

Theorem 2 is a consequence of Theorems 5.1 and 6.1 of [SAU]. The difficulty is proving that T satisfies Hypotheses (PE1) to (PE5).

To check that (PE2) is satisfied, we first prove that T_k is a C^1 diffeomorphism from W_k on $T_k(W_k)$. Hypothesis (H3) about $\frac{\partial \varphi_k}{\partial u}$ ensures that T_k is a local diffeomorphism. To establish the injectivity, let us consider two different points (x, y) and (x', y') of W_k , whose image under T is the same. One then has $y = y'$ and $\varphi_k(x', y/\gamma) = \varphi_k(x, y/\gamma)$. Using the geometrical hypothesis (H4) and applying the Mean Value Theorem to the application $t \mapsto \varphi_k(\Gamma_1(t), \Gamma_2(t))$, one obtains a contradiction.

The regularity hypotheses on φ_k (and consequently on T_k) imply that $\det(DT_k^{-1})$ is Hölder continuous for the exponent α , on a suitably restricted domain. One can prove that there exist, for every k , a real number $\beta_k > 0$, an open set \mathcal{V}_k with compact closure and a constant c_k such that

- $\overline{U_k} \subset \mathcal{V}_k \subset \overline{\mathcal{V}_k} \subset W_k$;
- $B_{\beta_k}(T_k(U_k)) \subset T_k(\mathcal{V}_k)$;
- for every $\varepsilon < \beta_k$, every $z \in T_k(\mathcal{V}_k)$ and all $x, y \in B_\varepsilon(z) \cap T_k(\mathcal{V}_k)$, one has

$$\left| \det(DT_k^{-1}(x)) - \det(DT_k^{-1}(y)) \right| \leq c_k \left| \det(DT_k^{-1}(z)) \right| \varepsilon^\alpha.$$

Setting $\beta = \min_k \beta_k > 0$ and $c = \max_k c_k > 0$, one gets constants which are valid for all $k \in \{1, \dots, d\}$. Hence (PE2) is satisfied.

This allows to fix the open set with which we are going to work: there exists $\varepsilon_2 > 0$ such that, for all $k \in \{1, \dots, d\}$, $B_{2\varepsilon_2}(\overline{U_k}) \subset \mathcal{V}_k \subset W_k$. From now on, $V_k = B_{\varepsilon_2}(\overline{U_k})$. The set

$T_k(V_k)$ is open and $T_k(\overline{U_k})$ is a compact set included in $T_k(V_k)$. One can find $\varepsilon_0^1 > 0$ such that $B_{\varepsilon_0^1}(T_k(\overline{U_k})) \subset T_k(V_k)$ for all k . Hypothesis (PE1) is thus verified.

Hypothesis (PE3) is clearly satisfied because $\Omega = \bigcup_{k=1}^d U_k \cup \mathcal{N}'$ is the disjoint union of open sets and of a negligible set.

One treats (PE4) in two steps : first one proves an expansion result, in the case when the arguments in \mathcal{V}_k are near (Proposition 5), then one proves (PE4) itself, which is an expansion result in the case when the images (in $T_k(V_k)$) are near.

Proposition 5 *Let (x, y) and $(x', y') \in \mathcal{V}_k$ be such that the segment $[(x, y), (x', y')]$ is included in \mathcal{V}_k . Then*

$$||T_k(x, y) - T_k(x', y')||^2 \geq \frac{1}{s^2} ||(x, y) - (x', y')||^2.$$

Proof: Applying the Mean Value Theorem to the application defined on $[0, 1]$ by $t \mapsto \varphi_k(x + t(x' - x), \frac{1}{\gamma}(y + t(y' - y)))$ gives a number $c \in]0, 1[$ such that

$$||T_k(x, y) - T_k(x', y')||^2 = (x' - x, y' - y) B \begin{pmatrix} x' - x \\ y' - y \end{pmatrix}$$

where

$$B = \begin{pmatrix} \gamma^2 \left(\frac{\partial \varphi_k}{\partial u}(x_c, \frac{1}{\gamma} y_c) \right)^2 & \gamma \frac{\partial \varphi_k}{\partial u}(x_c, \frac{1}{\gamma} y_c) \frac{\partial \varphi_k}{\partial v}(x_c, \frac{1}{\gamma} y_c) \\ \gamma \frac{\partial \varphi_k}{\partial u}(x_c, \frac{1}{\gamma} y_c) \frac{\partial \varphi_k}{\partial v}(x_c, \frac{1}{\gamma} y_c) & \frac{1}{\gamma^2} + \left(\frac{\partial \varphi_k}{\partial v}(x_c, \frac{1}{\gamma} y_c) \right)^2 \end{pmatrix}$$

with $(x_c, y_c) = (x + c(x' - x), y + c(y' - y))$.

The matrix B is real and symmetrical. Set

$$\begin{aligned} \xi_1 &= \text{Tr}(B) = \frac{1}{\gamma^2} + \left(\frac{\partial \varphi_k}{\partial v}(x_c, \frac{1}{\gamma} y_c) \right)^2 + \gamma^2 \left(\frac{\partial \varphi_k}{\partial u}(x_c, \frac{1}{\gamma} y_c) \right)^2 \\ \xi_2 &= \det(B) = \left(\frac{\partial \varphi_k}{\partial u}(x_c, \frac{1}{\gamma} y_c) \right)^2. \end{aligned}$$

We now prove that the eigenvalues of B are greater than $\frac{1}{s^2}$. Indeed, the map $\zeta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $\zeta(x, y) = (x + y, xy)$ is a bijection from

$$V_s'' = \{(x, y) \in \mathbb{R}^2 : s^{-2} \leq x \leq y\}$$

to

$$\zeta(V_s'') = \{(\xi_1, \xi_2) \in (\mathbb{R}_+^*)^2 : \xi_1 \geq 2s^{-2}, \xi_2 \geq s^{-2}(\xi_1 - s^{-2}), \xi_2 \leq \frac{\xi_1^2}{4}\}.$$

One just has to check that (ξ_1, ξ_2) is in $\zeta(V_s'')$ to obtain the result.

Now since B has real eigenvalues, the discriminant of its characteristic polynomial is non-negative. Consequently $4\xi_2 \leq \xi_1^2$. The conditions on A and M and the choice of s and γ imply that the other inequalities are satisfied.

It follows that eigenvalues of the matrix B are greater than or equal to s^{-2} . Hence $||T_k(x, y) - T_k(x', y')||^2 \geq \frac{1}{s^2} ||(x, y) - (x', y')||^2$, which completes the first step. \square

Compacity arguments prove that there exists $\varepsilon_0^2 > 0$ such that, for all $(x, y) \in \overline{V_k}$,

$$B_{\varepsilon_0^2}(T_k(x, y)) \subset T_k(B_{\varepsilon_2}(x, y)).$$

Proposition 6 *Set $\varepsilon_0 = \min(\varepsilon_0^1, \varepsilon_0^2) > 0$. Recall that $\overline{U_k} \subset V_k \subset \overline{V_k} \subset \mathcal{V}_k \subset W_k$. Then:*

- *For all $(u_1, v_1), (u_2, v_2) \in T_k(V_k)$ satisfying $d((u_1, v_1), (u_2, v_2)) < \varepsilon_0$, the following inequality holds:*

$$s^2 d((u_1, v_1), (u_2, v_2)) > d(T_k^{-1}(u_1, v_1), (T_k^{-1}(u_2, v_2))),$$

with $s^2 < 1$.

- $B_{\varepsilon_0}(T_k(\overline{U_k})) \subset T_k(V_k)$.

Proof : The second assertion comes from the fact that $\varepsilon_0 \leq \varepsilon_0^1$ and from what we have obtained in (PE1).

The first assertion implies Condition (PE4) of Saussol. To prove it, let $(u_1, v_1), (u_2, v_2) \in T_k(V_k)$ satisfy $d((u_1, v_1), (u_2, v_2)) < \varepsilon_0$. Let $(x, y) = T_k^{-1}(u_1, v_1)$ be in V_k . According to the preceding remark, as ε_0 is smaller than ε_0^2 ,

$$(u_2, v_2) \in B_{\varepsilon_0}(T_k(x, y)) \subset T_k(B_{\varepsilon_2}(x, y)).$$

Hence $(x', y') = T_k^{-1}(u_2, v_2) \in B_{\varepsilon_2}(x, y) \subset \mathcal{V}_k$. According to the Proposition 5,

$$d((u_1, v_1), (u_2, v_2))^2 = \|T_k(x, y) - T_k(x', y')\|^2 > \sigma \|(x, y) - (x', y')\|^2,$$

which proves the result. □

To conclude, Hypothesis (PE5) is a consequence of Lemma 2.1 of Saussol and of Hypothesis (H5).

Since the hypotheses (PE1) to (PE5) are verified, Theorem 5.1 of [SAU] implies the properties 1 to 5 of Theorem 2 about V_α and L_m^1 . But, if $f \in E_i$, f is equal to 0 on Ω^c and then f belongs to $L_m^1(\Omega)$ and to $V_\alpha(\Omega)$.

To prove the point 6, we apply Theorem 6.1 of [SAU] on every $W_{j,l}$, on which a power of T is mixing. Using the notations of Point 5 of Theorem 5.2 of [SAU], one obtains the existence of real constants $C > 0$ and $\rho \in]0, 1[$ such that, for all (j, l) satisfying $1 \leq j \leq \dim(E_1)$, $0 \leq l \leq L_j - 1$, for every function $f \in L_{\mu_{j,l}}^1(\Omega)$ and for every function $h \in V_\alpha(\Omega)$,

$$\left| \int_{\Omega} (f - \mu_{j,l}(f)) \circ T^{kL_j} h \, d\mu_{j,l} \right| \leq C \|f - \mu_{j,l}(f)\|_{L_{\mu_{j,l}}^1} \|h\|_{\alpha, L\rho^k}.$$

Let then h be in $V_\alpha(\Omega)$ and f be in $L_\mu^1(\Omega)$ (so that $f \in L_{\mu_{j,l}}^1(\Omega)$ for every j, l). Taking the smallest common multiple L' of the L_j and summing the above inequalities, with k replaced with $k \frac{L'}{L_j}$, one gets

$$\left| \int_{\Omega} f \circ T^{kL'} h \, d\mu - \sum_{j=1}^{\dim(E_1)} \sum_{l=0}^{L_j-1} \mu(W_{j,l}) \mu_{j,l}(f) \mu_{j,l}(h) \right| \leq C \|h\|_{\alpha, \Omega} \|f\|_{L_\mu^1} \rho^k.$$

Point 7 is a direct consequence of Point 6, since $\dim(E_1) = 1$ and $L_1 = 1$. This completes the proof of Theorem 2. \square

Now let us prove Theorem 4. If $\begin{pmatrix} X_0 \\ \gamma X_1 \end{pmatrix}$ has the distribution μ , then so does $\begin{pmatrix} X_k \\ \gamma X_{k+1} \end{pmatrix}$. If $f \in L^1_\mu(\Omega)$ and if $h \in V_\alpha(\Omega)$, we then have:

$$\left| E \left(f \begin{pmatrix} X_{kL'} \\ \gamma X_{kL'+1} \end{pmatrix} h \begin{pmatrix} X_0 \\ \gamma X_1 \end{pmatrix} \right) - \sum_{j=1}^{\dim(E_1)} \sum_{l=0}^{L_j-1} \mu(W_{j,l}) \mu_{j,l}(f) \mu_{j,l}(h) \right| \leq C \|f\|_{L^1_\mu} \|h\|_{\alpha,\Omega} \rho^k.$$

In order for $Tr H$ to belong to $V_\alpha(\Omega)$, it is sufficient and necessary that H belongs to $L^\infty([-L, L], m)$ and satisfies

$$\sup_{0 < \varepsilon < \varepsilon_0} \varepsilon^{-\alpha} \int_{[-L, L]} \text{Osc}(H,]x - \varepsilon, x + \varepsilon[\cap [-L, L]) \, dx < \infty.$$

Moreover,

$$\begin{aligned} \|Tr H\|_{\alpha,\Omega} &= 2\gamma L \sup_{0 < \varepsilon < \varepsilon_0} \varepsilon^{-\alpha} \int_{[-L, L]} \text{Osc}(H,]x - \varepsilon, x + \varepsilon[\cap [-L, L]) \, dx \\ &\quad + 16(1 + \gamma) L \varepsilon_0^{1-\alpha} \|H\|_{L^\infty_m([-L, L])} + 2\gamma L \|H\|_{L^1_m([-L, L])}. \end{aligned}$$

Thus if H satisfies these conditions and if F is such that $Tr F$ belongs to $L^1_\mu(\Omega)$, for example if F is measurable and bounded on $[-L, L]$, one has

$$\begin{aligned} &\left| E(F(X_{k \times L'}) H(X_0)) - \sum_{j=1}^{\dim(E_1)} \sum_{l=0}^{L_j-1} \mu(W_{j,l}) \mu_{j,l}(Tr F) \mu_{j,l}(Tr H) \right| \\ &\leq C \|Tr F\|_{L^1_\mu} \left(2\gamma L \sup_{0 < \varepsilon < \varepsilon_0} \varepsilon^{-\alpha} \int_{[-L, L]} \text{Osc}(H,]x - \varepsilon, x + \varepsilon[\cap [-L, L]) \, dx \right. \\ &\quad \left. + 16(1 + \gamma) L \varepsilon_0^{1-\alpha} \|H\|_{L^\infty_m([-L, L])} + 2\gamma L \|H\|_{L^1_m([-L, L])} \right) \rho^k. \end{aligned}$$

In particular, if H is the characteristic function of an interval and if F is the characteristic function of a Borel set, we obtain the first assertion of Theorem 4.

4 Examples

4.1 A nonlinear example

For all $k \in \mathbb{Z}$ we denote by f_k the polynomial function $f_k(x) = -\frac{71}{2}x^2 - 214x + k - \frac{1}{2}$. For all $-179 \leq k \leq 250$, one defines the open set O_k by :

$$O_k = \{(u, v) \in]-1, 1[^2 / f_k(u) < v < f_{k+1}(u)\}.$$

We consider the applications φ_k defined on $B_1(\overline{O_k})$ ($\varepsilon_1 = 1$) for all $-179 \leq k \leq 250$ by :

$$\varphi_k(u, v) = 2v - 2f_k(u) - 1.$$

One defines $\varphi : [-1, 1]^2 \rightarrow [-1, 1]$ almost everywhere by setting $\varphi|_{O_k} = \varphi_{k|_{O_k}}$ for all $-179 \leq k \leq 250$. We now make sure that these functions and open sets satisfy the conditions specified in Section 2.

The condition about the open sets is easily verified, since $[-1, 1]^2 \setminus \bigcup_{k=-179}^{250} O_k$ is the union of segments and parabolic arcs. Moreover, the maximal number of arcs meeting at one point is $Y = 3$.

The regularity conditions are satisfied, because the φ_k are smooth on $B_1(\overline{O_k})$. Set $\alpha = 1$. The partial derivatives satisfy the following inequalities: for all $-179 \leq k \leq 250$ and all $(u, v) \in B_1(\overline{O_k})$ one has:

$$\left| \frac{\partial \varphi_k}{\partial v}(u, v) \right| = 2 = M$$

and

$$\left| \frac{\partial \varphi_k}{\partial u}(u, v) \right| = 2|71u + 214| > 2(214 - 71(1 + 1)) = 144 = A > M + 1.$$

In this case, $\gamma = \frac{1}{12}$. A computation shows that $s \leq \frac{1}{10}$ and $\eta < 1$.

One sets $\Omega = [-1, 1] \times [-\frac{1}{12}, \frac{1}{12}]$ and for all $-179 \leq k \leq 250$ one defines the open sets

$$U_k = \{(x, y) \in \mathring{\Omega} : f_k(x) < 12y < f_{k+1}(x)\}.$$

We obtain the applications

$$T_k(x, y) = (12y, \frac{2}{12}(12y - f_k(x)) - \frac{1}{12}).$$

If $-177 \leq k \leq 248$, $T_k(U_k) = \mathring{\Omega}$ and T_k is a bijection from U_k on $\mathring{\Omega}$.

Otherwise, one can check that T_{-178} is a bijection from U_{-178} on $\Omega_1 \cup \Omega_2$, where Ω_1 is the open subset of $\mathring{\Omega}$ above the line having the equation $y = \frac{2x+1}{12}$, Ω_3 , the open subset under the line having the equation $y = \frac{2x-1}{12}$ and Ω_2 , the one between both lines. One has similar relations for $k = -177, 249$ and 250 and other subsets of Ω .

Finally, the simple version of the geometrical condition is satisfied (the open set contains the horizontal segment).

The transformation T therefore admits an invariant density h_* .

Let P be the Frobenius-Perron operator associated with T . One can prove that the constant functions are not invariant by P and consequently that h_* is not constant. Indeed, set

$$\psi_k(x, y) = (214)^2 - 71(2x - 12y) + 142k.$$

Then Ph can be written as $Ph(x, y) = \sum_{k=a}^b h(T_k^{-1}(x, y)) \frac{1}{2\sqrt{\Psi_k(x, y)}}$, with $(a, b) = (-179, 248)$

if $(x, y) \in \Omega_1$, $(a, b) = (-178, 249)$ si $(x, y) \in \Omega_2$, $(a, b) = (-177, 250)$ if $(x, y) \in \Omega_3$.

We now verify that $P1 \neq 1$. Suppose that $h = 1$ and set $z = x - 6y$.

If $(x, y) \in \Omega_3$, $z \in]-\frac{3}{2}, -\frac{1}{2}[$. The function $z \mapsto \sqrt{(214)^2 - 142z + 142k}$ is strictly decreasing on $] -\frac{3}{2}, -\frac{1}{2}[$. Therefore $z \mapsto \frac{1}{2\sqrt{(214)^2 - 71(2x - 12y) + 142k}}$ is strictly increasing on $] -\frac{3}{2}, -\frac{1}{2}[$ and $P1$ is not constant.

4.2 A piecewise linear example

This example can be useful to create a generator of pseudo random numbers in $[-L, L]$. In this section, a and b are positive or negative integers, L is a positive integer or half integer.

One denotes by \mathcal{U}^2 the square $[-L, L]^2$. For all $n \in \mathbb{Z}$, the open set Ω_n is defined by

$$\Omega_n = \{(u, v) \in]-L, L[^2 : av + bu \in](2n-1)L, (2n+1)L[\}.$$

One denotes by Δ_n the line having the equation $av + bu = (2n-1)L$. One defines φ_n on \mathbb{R}^2 by

$$\varphi_n(u, v) = av + bu - 2nL.$$

Then $\varphi_n|_{\Omega_n}$ is valued in $] -L, L[$ and we set

$$\forall (u, v) \in \Omega_n, \varphi(u, v) = \varphi_n(u, v).$$

We impose the following condition, with $S = 1 + \frac{48}{\pi} + \frac{288}{\pi^2} + \frac{4}{\pi} \left(1 + \frac{12}{\pi}\right) \sqrt{6\pi + 36}$,

$$|a| < \frac{|b| - S}{\sqrt{S}}.$$

One verifies that the conditions of Section 2 are fulfilled.

The square \mathcal{U}^2 is the disjoint union of a finite number of open sets Ω_n and of a negligible set composed of a finite number of segments.

The maximal number of these segments meeting at one point is $Y = 3$.

For every $\eta > 0$, the open sets $B_\eta(\Omega_n)$ are convex, hence contain the horizontal segment joining two points having the same ordinate and the geometrical condition is satisfied.

The applications φ_n are smooth on $B_\eta(\Omega_n)$. We set $\alpha = 1$. Moreover $\varphi_n(\Omega_n) \subset [-L, L]$.

We set $M = |a|$, $A = |b|$, so that the partial derivatives satisfy the required inequalities. The upper bound of $|a|$ implies that $0 < M < A - 1$.

One sets $\gamma = |b|^{-1/2}$ (it is the compression coefficient) and one checks that $\eta < 1$.

We determine for which integers n the line Δ_n crosses the square. One can see that $\Delta_n \cap \mathcal{U}^2 \neq \emptyset$ if and only if

$$\frac{-|a| - |b| + 1}{2} \leq n \leq \frac{|a| + |b| + 1}{2}.$$

One defines $\mathcal{N}(a, b)$ as the set of indices n such that a nonempty Ω_n intersects \mathcal{U}^2 .

Set

$$\Omega = [-L, L] \times [-\gamma L, \gamma L]$$

and

$$\Omega_{n,a} = \{(x, y) \in \mathring{\Omega} : a\sqrt{|b|}y + bx - 2nL \in]-L, L[\}.$$

Set

$$\begin{aligned} T_n : \Omega_{n,a} &\rightarrow \Omega \\ (x, y) &\mapsto (\sqrt{|b|}y, ay + \frac{b}{\sqrt{|b|}}x - \frac{2nL}{\sqrt{|b|}}). \end{aligned}$$

One defines T almost everywhere from Ω in Ω by setting $T|_{\Omega_{n,a}} = T_n$.

The Frobenius-Perron operator P associated with T has the following expression:

$$Ph(x, y) = \frac{1}{|b|} \sum_{n \in \mathcal{N}(a, b)} \mathbf{1}_{(x, y) \in T_n(\Omega_{n, a})} h(T_n^{-1}(x, y)).$$

If h is a constant function equal to $c > 0$, one deduces that

$$Ph(x, y) = \frac{c}{|b|} \#\{n \in \mathcal{N}(a, b) : (x, y) \in T_n(\Omega_{n, a})\}.$$

One can see that this cardinal number is $|b|$, which proves that there exists a constant invariant density.

But the theorem proves the existence of an invariant measure h^*m , given by $P_1 \mathbf{1}_\Omega = h^*$. According to Lemma 4.1 of [ITM],

$$P_1 \mathbf{1}_\Omega = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=1}^n P^k \mathbf{1}_\Omega = \mathbf{1}_\Omega.$$

Consequently, h^* is a constant function.

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